J Stat Phys (2010) 141: 957–969
DOI 10.1007/s10955-010-0091-9

Ferromagnetism of Electron Gas

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Received: 11 November 2009 / Accepted: 1 November 2010 / Published online: 12 November 2010

Abstract We here investigate the density and temperature dependence of polarization using the relativistic formalism for the electron–electron interaction within the Fermi liquid model. The variational method has been used: the free energy has been minimized with respect to the effective mass and the polarization parameter. Then we obtained the equation of state and magnetic susceptibility of the system. The exact results for polarization and magnetic susceptibility have been obtained at zero temperature. It has been shown that for a given temperature (density) there is a critical density (temperature) at which the ferromagnetic phase can appear in an electron gas. The results are in agreement with previous work. Our results show that at nonzero temperatures and in very low and very high densities the ferromagnetism phase cannot exist.

Keywords Ferromagnetism · Landau-Fermi liquid · Free energy · Magnetic susceptibility · Variational method

1 Introduction

The properties of Fermi systems have been investigated in several works [1–25]. One of the important cases is the study of the magnetic properties of an electron gas [4–25]. Spontaneous magnetization may appear at different densities for different temperatures and the polarization of the system is a function of density and temperature. By assuming a spin–spin interaction inside the system, we can study the magnetic properties of the system. For gaseous systems, statistical methods for an imperfect Fermi gas show that the system can be in its ferromagnetic phase [26]. Here we examine the possibility of the existence of the ferromagnetism phase inside a Fermi gas with one photon exchange interaction and the variational method. Because the spin dependence of the Landau–Fermi liquid interaction function is due to relativistic effects and exchange interactions [27–29], we use the

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relativistic formalism to calculate the direct and exchange diagrams' contributions to thermodynamic quantities. All relations are written in the non-relativistic limit, because in the ultra-relativistic region ($m \ll k_f$) the polarized states have greater energy than the unpolarized states [29, 30]. Therefore, the ferromagnetic phase can appear in the non-relativistic region [30]. The resultant scattering matrix elements in the non-relativistic region automatically have spin dependent terms and we do not insert the spin–spin interaction manually [31]. We use the variational approach [32–41] to obtain the equation of state of the system. By varying the free energy with respect to p (the polarization parameter) and effective mass at various densities and temperatures, we can minimize the free energy for a given density and temperature. Finally, the magnetic susceptibility of the system at zero temperature has been obtained. Our results show that χ has a divergence at $k_f \to \frac{m\alpha}{\pi}$. This means that the phase transition is second order inside the system. Therefore, the paper develops as follows: in Sect. 2 we review the exchange and direct interactions and the equation of state at zero temperature. The equation of state at low temperature is expressed in Sect. 3. Results and discussion are given in Sect. 4 and a summary is given in Sect. 5.

2 Exchange and Direct Contributions in Energy Density

To obtain the exchange and direct diagrams contributions we write the Lorentz invariant matrix elements for two electron scattering in order g^2. In the presence of a positive charged background, one can show that the direct diagram portion of the energy density is [42, 43]:

$$E_d \sim -\rho V^{\frac{2}{3}}. \tag{1}$$

The $\frac{E_d}{N} \sim V^{\frac{-1}{3}}$ becomes zero in the thermodynamic limit. Therefore, we can ignore the direct interaction and only calculate the exchange interaction contribution to the thermodynamic limit. We define the up and down spin states in the rest frame of each electron, with the vector $\vec{\xi} = (0, 0, \pm 1)$, where the up and down spin states corresponded to the $\pm$ signs. Electrons are described by a wave function $u(p, s)$, where p is the electron momentum and s is the electron spin four vector. If the electron is at rest, s^μ reduces to $\vec{\xi} = (0, 0, \pm 1)$. In the frame in which the particle moves with momentum $\vec{p}$, the components of s^μ are obtained by Lorentz transformation [28, 44]:

$$s^\mu = \left(\frac{\vec{p}.\vec{\xi}}{m}, \vec{\xi} + \frac{\vec{p}.\vec{\xi}}{m(E+m)} \vec{p} \right). \tag{2}$$

Where $E = \sqrt{m^2 + p^2}$. The state with definite polarization is obtained by applying the projection operator $\Sigma(s) = \frac{1}{2}(1 + \gamma_5 \not{s})$. The polarization density matrix can be written as [28]:

$$\rho(p, s) = \frac{1}{2m}(\not{p} + m)\Sigma(s). \tag{3}$$

For the exchange interaction we can write the Lorentz invariant matrix elements as follows [28]:

$$\mathcal{M}^s_{\mathbf{k}\xi, \mathbf{q}\xi'} = g^2 \mathrm{tr}(\gamma_\mu \rho(k, \xi) \gamma^\mu \rho(q, \xi')) \frac{1}{(k-q)^2}. \tag{4}$$

It must be considered that when we use the above relation in Landau theory, the stability of the Fermi liquid is not satisfied. If we add a term like δ^2 in the denominator of (4), then stability can be satisfied. The δ^2 can interpreted as higher order corrections to the gauge boson propagator. For simplicity we ignore the δ^2 and set it equal to zero. After taking traces the result becomes [28]:

$$\begin{aligned}\mathcal{M}^{s}_{\mathbf{k}\xi,\mathbf{q}\xi'} = \frac{1}{2}\frac{g^2}{m^2}\bigg[&2m_q^2 - k.q - (\mathbf{k}.\xi)(\mathbf{q}.\xi') + m^2\xi.\xi' + \frac{1}{(\varepsilon_k+m)(\varepsilon_q+m)}\\ &\times\{m(\varepsilon_k+m)(\xi.\mathbf{q})(\xi'.\mathbf{q}) + m(\varepsilon_q+m)(\xi.\mathbf{k})(\xi'.\mathbf{k})\\ &+ (\mathbf{k.q})(\xi.\mathbf{k})(\xi'.\mathbf{q})\}\bigg]\frac{1}{(k-q)^2}.\end{aligned} \tag{5}$$

The Landau–Fermi liquid interaction function is related to the Lorentz invariant matrix element via:

$$f_{k\xi,q\acute{\xi}} = \frac{m^2}{E_k E_q}\mathcal{M}^{s}_{\mathbf{k}\xi,\mathbf{q}\xi'}. \tag{6}$$

If $\xi = \xi'$ (parallel spins), we have the spin non-flip interaction and if $\xi = -\xi'$ (anti parallel spins), we have flip interaction. So the exchange energy density for flip and non-flip interactions can be written:

$$\varepsilon_{ex}^{flip} = \iint \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} n(k^+)n(q^-) f^{flip}_{\mathbf{k},\mathbf{q}}, \tag{7}$$

$$\varepsilon_{ex}^{non\text{-}flip} = \frac{1}{2}\sum_{i=\pm}\iint \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} n(k^i)n(q^i) f^{non\text{-}flip}_{\mathbf{k},\mathbf{q}}. \tag{8}$$

In above equations ± correspond to

$$\begin{aligned} n_+ &= n_q(1+p)/2,\\ n_- &= n_q(1-p)/2.\end{aligned} \tag{9}$$

Where the $n_\pm$ and p are the density of spin up and spin down electrons and polarization parameter, respectively. $n(k^i)$ are the Fermi distribution functions. The integrals can be calculated over angles using following relations [45]:

$$\begin{aligned} &\int \frac{d\Omega}{1+\vec{k}.\hat{r}} = \frac{2\pi}{k}\ln\left(\frac{1+k}{1-k}\right),\\ &\int d\Omega(\hat{r}.\vec{a})(\hat{r}.\vec{b}) = \frac{4\pi}{3}\vec{a}.\vec{b},\\ &\int d\Omega\frac{\vec{a}.\hat{r}}{1+\vec{k}.\hat{r}} = \frac{4\pi}{k^2}\vec{a}.\vec{k}\left[1-\frac{1}{2k}\ln\left(\frac{1+k}{1-k}\right)\right],\\ &\int d\Omega\frac{(\vec{a}.\hat{r})(\vec{b}.\hat{r})}{1+\vec{a}.\hat{r}} = \frac{2\pi}{a^3}\ln\left(\frac{1+k}{1-k}\right)\vec{a}.\vec{b}.\end{aligned} \tag{10}$$

After integrating over angles, we perform the integration on the momenta by numerical methods. The numerical integration arguments are as follow [30]:

$$\begin{aligned}
\varepsilon_{ex}^{flip} &= \frac{1}{(2\pi)^6}\int n(k^+)n(q^-)k^2q^2dkdq\frac{A(k,q)}{B(k,q)}\left(\sum_{i=1}^{6}T_i\right),\\
\varepsilon_{ex}^{non\text{-}flip} &= \frac{1}{2}\frac{1}{(2\pi)^6}\sum_{i=\pm}\int n(k^i)n(q^i)k^2q^2dkdq\frac{A(k,q)}{B(k,q)}\left(\sum_{i=1}^{6}T_i\right),
\end{aligned} \tag{11}$$

with:

$$\begin{aligned}
A(k,q) &= \frac{m^2}{\varepsilon_k\varepsilon_q}\frac{2g^2}{18m^2kq},\\
B(k,q) &= \frac{m^2-\varepsilon_k\varepsilon_q}{kq},\\
B_1(k,q) &= 2m^2+m^2\vec{\zeta}.\vec{\zeta}'-\varepsilon_k\varepsilon_q,\\
Z_1(k,q) &= \frac{m(\varepsilon_k+m)}{(\varepsilon_k+m)(\varepsilon_q+m)},\\
Z_2(k,q) &= \frac{m(\varepsilon_q+m)}{(\varepsilon_k+m)(\varepsilon_q+m)},\\
Z_3(k,q) &= \frac{1}{(\varepsilon_k+m)(\varepsilon_q+m)},\\
T_1 &= 8\pi^2B_1(k,q)\ln\left(\frac{B(k,q)+1}{B(k,q)-1}\right),\\
T_2 &= 16\pi^2kq\left(1-\frac{B(k,q)}{2}\ln\left(\frac{B(k,q)+1}{B(k,q)-1}\right)\right),\\
T_3 &= \frac{-16\pi^2}{3}kq\left(1-\frac{B(k,q)}{2}\ln\left(\frac{B(k,q)+1}{B(k,q)-1}\right)\right)\vec{\zeta}.\vec{\zeta}',\\
T_4 &= \frac{8\pi^2}{3}Z_1(k,q)q^2\ln\left(\frac{B(k,q)+1}{B(k,q)-1}\right)\vec{\zeta}.\vec{\zeta}',\\
T_5 &= \frac{8\pi^2}{3}Z_2(k,q)k^2\ln\left(\frac{B(k,q)+1}{B(k,q)-1}\right)\vec{\zeta}.\vec{\zeta}',\\
T_6 &= Z_3(k,q)\frac{8\pi^2}{3}B(k,q)^2k^2q^2\ln\left(\frac{B(k,q)+1}{B(k,q)-1}\right)\vec{\zeta}.\vec{\zeta}'.
\end{aligned} \tag{12}$$

The integration results for $p=0$ at zero temperature are the same as the unpolarized exchange energy results obtained before [29, 42, 46].

$$\varepsilon_{ex}^{unpol} = -\frac{\alpha}{4\pi^3}\left\{k_f^4-\frac{3}{2}\left[E_fk_f-m^2\ln\left(\frac{E_f+k_f}{m}\right)\right]^2\right\}, \tag{13}$$

where $E_f = \sqrt{m^2 + k_f^2}$ is the Fermi energy and $\alpha = g^2/4\pi$. According to Fermi liquid theory the interaction function on the Fermi surface has the form [27]:

$$\left(\frac{k_f^2}{\pi^2 v_f}\right) f_{k\xi,q\acute{\xi}} = F(\theta) + \vec{s}.\vec{s}' G(\theta). \tag{14}$$

In (14) θ is the angle between the two electron momenta on the Fermi surface. Comparing (14) with (6) and (12), one can see that on the Fermi surface only the scalar product of two electron spin operators appears [27]. In the non-relativistic region we can use $k_f \ll m$ approximation and then we have [28]:

$$\mathcal{M}_{non\text{-}rel} = \frac{-g^2}{2} \frac{1 + \vec{\xi}.\vec{\xi}'}{|\vec{k} - \vec{q}|^2}. \tag{15}$$

According to (15) if we have $\xi = -\xi'$ then the Lorentz invariant matrix elements vanish, this means that in the non-relativistic region the spin flip contribution in energy density vanishes. The spin non-flip exchange and kinetic energy density in the non-relativistic case at zero temperature is:

$$\begin{aligned} \varepsilon_{ex}^0 &= \frac{-\alpha}{8\pi^3} k_f^4 ((1+p)^{\frac{4}{3}} + (1-p)^{\frac{4}{3}}), \\ \varepsilon_{kin}^0 &= \frac{k_f^5}{20m\pi^2}((1+p)^{\frac{5}{3}} + (1-p)^{\frac{5}{3}}). \end{aligned} \tag{16}$$

In the ultra-relativistic region ($k_f \gg m$) the exchange energy density is proportional to $\alpha k_f^4((1+p)^{\frac{4}{3}} + (1-p)^{\frac{4}{3}})$ [29] and the kinetic energy is also proportional to $k_f^4((1+p)^{\frac{4}{3}} + (1-p)^{\frac{4}{3}})$. Therefore, the unpolarized state is energetically favorable and we will write all equations in the non-relativistic limit.

3 Equation of State at Low Temperature

To obtain the equation of sate at low temperature we use the variational method with the following approximation to the single particle energy at the Fermi distribution function [2, 32, 40, 41]:

$$\begin{aligned} n(k) &= \frac{1}{\exp\frac{E-\mu}{T} + 1}, \\ E &\approx \frac{k^2}{2m^*(n,T)}. \end{aligned} \tag{17}$$

According to the above approximation we use the redefined kinetic energy against the sum of kinetic and potential energy in distribution function, and the kinetic energy in other parts of the equations remains unchanged [2, 32, 38, 40, 41]. For a fully polarized state ($p = 1$) and an unpolarized state ($p = 0$), we can describe system with a single effective mass and chemical potential. For partially polarized sates ($0 < p < 1$) we must use separate effective mass and chemical potentials for spin up and spin down states. However, at non-zero and low temperatures, the partially polarized states appear only within a narrow region of density and temperature. Therefore, we may use an approximation that assumes that both spin up and down states have the same effective mass. In the zero temperature case, the effective mass does not appear in the equations, because the Fermi distribution reduces to a simple step function. Therefore, the zero temperature results are independent of our approximation. The system is in chemical equilibrium and the total chemical potential of the system is the sum of the chemical potentials of the spin up and spin down states, because there is no interaction between the different spin states inside the system (see (15)). By minimizing the free energy with respect to the variational parameter m^* we can obtain the equation of state. The physical meaning of effective mass is similar to its meaning in nuclear matter [43] and Landau theory, but the definition of effective mass is different in the variational approach. In the variational method, we define the effective mass as a parameter that minimizes the thermodynamic potential (free energy) and similar to nuclear matter and Landau theory is a function of density and temperature. The effective mass contains the information about the single particle interaction potential of the system. In our approximation m^* is the variational parameter that minimizes the whole system's free energy and is not for different states inside the system. To understand the notion of the variational method used, we assume the single particle Hamiltonian under our approximation

$$H = \frac{p^2}{2m^*}. \tag{18}$$

If we use quantum mechanics, we can find the following relation for the mean group velocity of particles:

$$\langle v \rangle = \frac{p}{m^*}. \tag{19}$$

This means that the mean group velocity of particles depends on $m^*(n, T)$. On the other hand, the mean value of operators depends on the shape of wave function of particles. In this manner, the variation of effective mass corresponds to variation of wave function of single particles inside system or more generally the variation of the total system's wave function. In other words if we represent the wave function with time evolution operator:

$$|\alpha, t\rangle = u(t, t_0)|\alpha, t_0\rangle, \tag{20}$$

then the effect of H clearly appears in the definition of the wave function. We must mention that because of the density and temperature dependence of Hamiltonian, the effective mass also is a function of density and temperature. By varying the effective mass, we find a suitable wave function that satisfies the second law of thermodynamics. If we have $ds \geq 0$ and $ds = 0$ in equilibrium state, then we also can show that $dF \leq 0$ and $dF = 0$. Therefore, minimizing the free energy with respect to effective mass corresponds to finding an appropriate wave function that satisfies the second law of thermodynamics. If we solve the equations by normal statistical mechanics, then the $dF = 0$ is automatically fulfilled. But in our approximation, $E_k + V \approx \frac{p^2}{2m^*}$, we replace an interacting system with an ideal system. This approximation is valid only for momentum dependent interactions (i.e., exchange interaction). If m^* is constant then again we have $dF = 0$. If m^* is not constant ($m^* = m^*(N, T, V)$) then $dF = 0$ must be satisfied. The other relations like $dS = 0$ or $dG = 0$ (Gibbs free energy), don't give us independent relations. We multiply $\frac{p^2}{2m}$ by the number of states and integrate over all momenta to obtain the kinetic energy [48]:

$$\frac{E_{kin}}{V} = \frac{4\pi}{(2\pi)^3} \int \frac{\frac{p^2}{2m} p^2 dp}{\exp(\beta(\frac{p^2}{2m^*} - \mu)) + 1}. \tag{21}$$

If we use the following relation:

$$\int \frac{k^s dk}{\exp(k-\mu)+1} = e^{\mu}\Gamma(s+1)\Phi(-e^{\mu}, s+1, 1), \tag{22}$$

(where the Φ is Lerch Phi function) we can write the kinetic energy as follows:

$$\frac{E_{kin}}{V} = \frac{4\pi}{(2\pi)^3 2m} 2^{\frac{5}{2}} \left(\frac{m^*}{\beta}\right)^{\frac{3}{2}} e^{\mu}\Gamma\left(\frac{5}{2}\right)\Phi\left(-e^{\mu}, \frac{5}{2}, 1\right). \tag{23}$$

The chemical potential is a function of effective mass also. We can obtain the chemical potential by solving the following equation:

$$\frac{N}{V} = \int \frac{\frac{d_3 p}{(2\pi)^3}}{\exp(\beta(\frac{p^2}{2m^*} - \mu)) + 1}. \tag{24}$$

The appearance of m and effective mass in kinetic energy term comes from the definition of single particle kinetic energy (we approximate the total single particle energy not kinetic energy and potential energy separately). Similar relations can be written for the potential energy term and we can solve equation $dF = 0$ and obtain the effective mass. At zero temperature we can define the effective mass ($\lim_{T\to 0} m^*(n, T)$), but it is convenient to write equations with respect to the other variational parameters, like the polarization parameter or the Fermi momentum, Because at zero temperature the Fermi distribution function reduces to a step function and the effective mass does not appear expressly in the distribution function. To obtain the kinetic energy at low temperature we used the following relations:

$$\begin{aligned}\frac{E^{\pm}_{kin}(T)}{V} &= \frac{4\pi}{(2\pi)^3 2m} \int \frac{p^4 dp}{\exp\beta(\frac{p^2}{2m^*} - \mu_{(\pm)}) + 1} \\ &= \frac{m^*(2m^*)^{\frac{3}{2}}}{4\pi^2 m} \frac{2}{5} \mu^{\frac{5}{2}}_{(\pm)} \left[1 + \frac{5\pi^2 T^2}{8} \mu^{-2}_{(\pm)}\right].\end{aligned} \tag{25}$$

Where ± refers to spin up and down states. If we use the low temperature expansion of the chemical potential, then we have:

$$\varepsilon^{non\text{-}rel}_{kin}(T) = \varepsilon^0_{kin}\left[1 + \frac{5\pi^2}{12}\left(\frac{2m^*T}{k_f^2}\right)^2\right]. \tag{26}$$

In (26), $\varepsilon^{non\text{-}rel}_{kin}(T)$ is the kinetic energy density of the system. Similar to the above, the exchange energy and entropy density of the system become:

$$\begin{aligned}\varepsilon^{non\text{-}rel}_{ex}(T) &= \varepsilon^0_{ex}\left[1 - \frac{\pi^2}{6}\left(\frac{2m^*T}{k_f^2}\right)^2\right], \\ S = s^+ + s^- &= \frac{\pi^2 n m^* T}{2k_f^2}[(1+p)^{\frac{1}{3}} + (1-p)^{\frac{1}{3}}].\end{aligned} \tag{27}$$

Where m^*, ε^0_{kin} and ε^0_{ex} are the effective mass of the electrons, non-relativistic kinetic and exchange energies at zero temperature, respectively. Using the results of (27) one can obtain the following result for the free energy density:

$$\begin{aligned} F &= \varepsilon_{ex} + \varepsilon_{kin} - TS \\ &= \varepsilon_{ex}^0\left[1 - \frac{\pi^2}{6}\left(\frac{2m^*T}{k_f^2}\right)^2\right] + \varepsilon_{kin}^0\left[1 + \frac{5\pi^2}{12}\left(\frac{2m^*T}{k_f^2}\right)^2\right] \\ &\quad - \frac{\pi^2 nm^*T}{2k_f^2}[(1+p)^{\frac{1}{3}} + (1-p)^{\frac{1}{3}}]. \end{aligned} \tag{28}$$

According to the fundamental relations of thermodynamics, the free energy must be a minimum at the equilibrium state, so we must have:

$$dF|_{T,n} = 0. \tag{29}$$

At fixed density and temperature, we can write:

$$\frac{\partial F}{\partial m^*}dm^* + \frac{\partial F}{\partial p}dp = 0. \tag{30}$$

Because the effective mass and polarization parameter are the independent variables for free energy, then we must have:

$$\begin{aligned} \frac{\partial F}{\partial m^*} &= 0, \\ \frac{\partial F}{\partial p} &= 0. \end{aligned} \tag{31}$$

Solving the above equations simultaneously, we can obtain the density and temperature dependence of m^* and p. The derivatives of free energy with respect to p and m^* are as follows:

$$\begin{aligned} \frac{\partial F}{\partial m^*} &= \left(\frac{T}{k_f^2}\right)^2 m^*\left[\varepsilon_{kin}^0\left(\frac{10\pi^2}{3}\right) - \varepsilon_{ex}^0\frac{4\pi^2}{3}\right] \\ &\quad - \left(\frac{T}{k_f}\right)^2\frac{\pi^2 n}{2}[(1+p)^{\frac{1}{3}} + (1-p)^{\frac{1}{3}}], \end{aligned} \tag{32}$$

$$\begin{aligned} \frac{\partial F}{\partial p} &= \left[1 + \frac{5\pi^2}{12}\left(\frac{2m^*T^2}{k_f}\right)\right]\frac{\partial \varepsilon_{kin}^0}{\partial p} + \left[1 - \frac{\pi^2}{6}\left(\frac{2m^*T}{k_f}\right)^2\right]\frac{\partial \varepsilon_{ex}^0}{\partial p} \\ &\quad - \frac{\pi^2 nm^*}{6}\left(\frac{T}{k_f}\right)^2[(1+p)^{\frac{1}{3}} - (1-p)^{\frac{1}{3}}]. \end{aligned} \tag{33}$$

The derivatives of kinetic and exchange energies with respect to p are given below:

$$\frac{\partial}{\partial p}\varepsilon_{ex}^0 = -\frac{4}{3}\frac{\alpha}{8\pi^3}k_f^4[(1+p)^{\frac{1}{3}} - (1-p)^{\frac{1}{3}}], \tag{34}$$

$$\frac{\partial}{\partial p}\varepsilon_{kin}^0 = \frac{5}{3}\frac{k_f^5}{20\pi^2 m}[(1+p)^{\frac{2}{3}} - (1-p)^{\frac{2}{3}}]. \tag{35}$$

By solving (32), one can find m^* as a function of p:

$$m^* = \frac{3k_f^2 n}{4} \frac{(1+p)^{\frac{1}{3}} + (1-p)^{\frac{1}{3}}}{5\varepsilon^0_{kin} - 2\varepsilon^0_{ex}}. \tag{36}$$

For $p = 0$ and non-interacting systems (36) simply yields $m^* = m$. We must notice here that if we set $\alpha = 0$ (non-interacting system), (31) gives $p = 0$. In general, where the system is partially polarized, we rewrite (36) as follows:

$$\frac{k_f^2}{2m^*} = \frac{2}{3n} \frac{5\varepsilon^0_{kin} - 2\varepsilon^0_{ex}}{(1+p)^{\frac{1}{3}} + (1-p)^{\frac{1}{3}}}. \tag{37}$$

Because $\varepsilon^0_{ex} \propto k_f^5$ and $n \propto k_f^3$,

$$\frac{1}{m^*} \propto \left(\frac{A}{m} + B\varepsilon^0_{ex}\right), \tag{38}$$

where A and B are functions of polarization and density. Equation (38) shows that the effective mass directly depends on interaction potential of the system (the constant A becomes a unit when the interaction is zero). At zero temperature we can use the results of (16) and rewrite (38) as follows:

$$\frac{k_f^2}{2m^*} = \frac{k_f^2}{2m}\left[\frac{(1+p)^{\frac{5}{3}} + (1-p)^{\frac{5}{3}}}{(1+p)^{\frac{1}{3}} + (1-p)^{\frac{1}{3}}}\right] + \frac{\alpha k_f^2}{2\pi}\left[\frac{(1+p)^{\frac{4}{3}} + (1-p)^{\frac{4}{3}}}{(1+p)^{\frac{1}{3}} + (1-p)^{\frac{1}{3}}}\right]. \tag{39}$$

If we set $\alpha = 0$ then the polarization parameter becomes zero (see (41)). So for non-interacting systems we find $m^* = m$. Inserting the result of (36) in (33), we can obtain the density and temperature dependence of polarization parameter. At zero temperature, (33) becomes simpler and one obtains

$$\frac{\partial}{\partial p}\varepsilon^0_{ex} + \frac{\partial}{\partial p}\varepsilon^0_{kin} = 0. \tag{40}$$

The resulting equation from (40) is:

$$\frac{k_f}{2m}[(1+p)^{\frac{2}{3}} - (1-p)^{\frac{2}{3}}] = \frac{\alpha}{\pi}[(1+p)^{\frac{1}{3}} - (1-p)^{\frac{1}{3}}]. \tag{41}$$

This result is very similar to the well-known results of the spontaneous magnetization of an imperfect Fermi gas [26]. But in our calculation, the interaction part of the Hamiltonian is not independent of the spin alignment, and this dependence changes the right side of (41).

4 Results and Discussion

At zero temperature we can solve (33). The result is shown in Fig. 1, there is a specific density (dependent on the coupling constant), such that for densities higher than it, the ferromagnetism phase will appear. This figure also shows that if the density becomes small, the ferromagnetism phase cannot occur. At a non-zero temperature we can solve (32) and find m^* as a function of p, T and the density. Minimizing the free energy with respect to p,

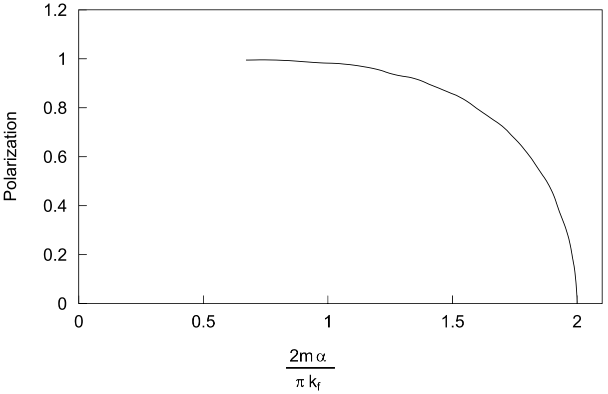
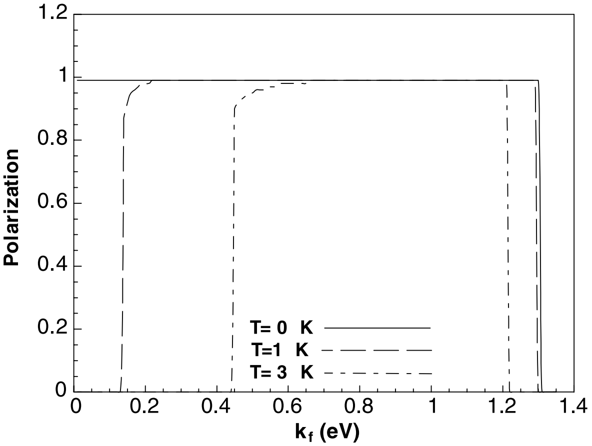


Fig. 1 The polarization parameter of system at zero temperature. The horizontal axis shows $\frac{2\alpha m}{\pi k_f}$. For $\alpha m = \pi k_f$ the polarization of the system vanishes. The value of the critical density for phase transition is proportional to the coupling constant. If we ignore the interaction inside system ferromagnetism cannot appear and if we use the effective and great coupling constant then the critical density will increase

Fig. 2 The density dependence of polarization parameter at various temperatures. By increasing the temperature, the ferromagnetism bound of the system become narrower. So we expect that for specific temperature (T_α) the ferromagnetic phase completely disappears and the system cannot be in ferromagnetic phase at any density. The T_α strongly depends on interactions inside the system or in other words depends strongly on the coupling constant



at constant temperature (density) yields Fig. 2, resp., Fig. 3. Figure 2 shows the density dependence of the polarization parameter at various temperatures. Increasing the temperature leads to the disappearance of the ferromagnetic phase. At low densities, also the ferromagnetism disappears. Comparing the Fermi energy of the system to the thermal energy of the system can help us to find the reason. For states that have a Fermi energy smaller than their thermal energy, the Fermi distribution function reduces to the Maxwell distribution and the difference between different spin states is ignorable. Figure 3 displays the temperature dependence of the polarization parameter at different densities. As we expected, by increasing the temperature, the polarization of the system reduces and suddenly vanishes. To obtain the magnetic susceptibility of the system we can use the following relation [47]:

$$\chi^{-1} = -\frac{1}{n^2\mu_B^2}\frac{d^2F}{dp^2}\bigg|_{p=0}. \tag{42}$$

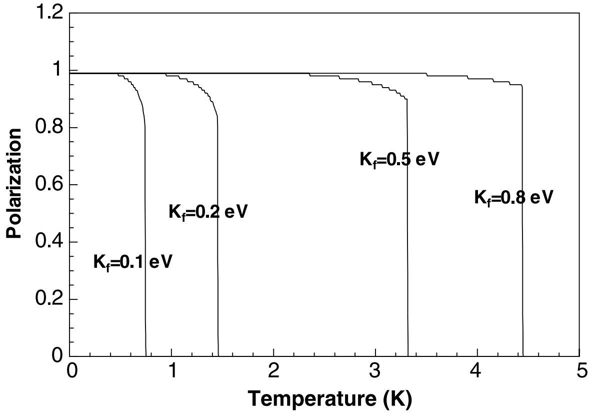


Fig. 3 The polarization parameter as a function of temperature at different densities. By increasing the temperature, the polarization of the system is reduced

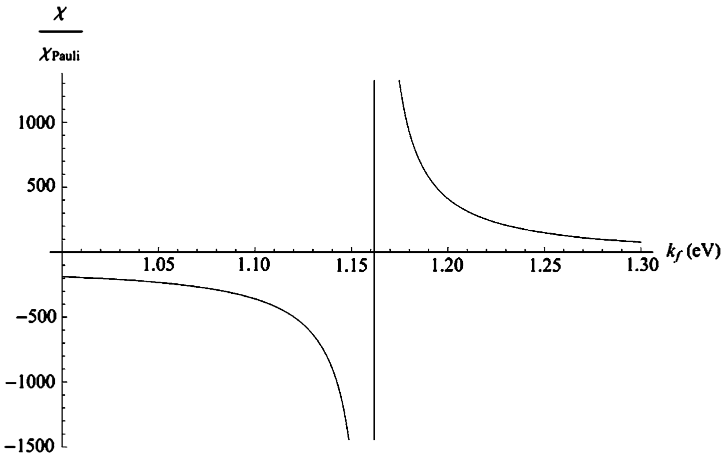


Fig. 4 The $\frac{\chi}{\chi_{Pauli}}$ at zero temperature as a function of Fermi momentum of system. As $k_f \longrightarrow \frac{m\alpha}{\pi}$ the χ tends to infinity and diverges. This means that the phase transition is of second order

At zero temperature, $p = 0$ is equivalent to $k_f = \frac{m\alpha}{\pi}$. The magnetic susceptibility of the system relative to χ_{Pauli} ($\chi_{Pauli} = \frac{\mu_B^2 k_f m}{\pi^3}$ [48]) is plotted as a function of Fermi momentum in Fig. 4. It has a divergence at $k_f = \frac{m\alpha}{\pi}$. This divergence is a sign of a second order phase transition inside the system. By definition of the dimensionless parameter $r_s = (\frac{9\pi}{4})^{\frac{1}{3}} \frac{\alpha m}{k_f}$ [49], one can see that the magnetic susceptibility divergence appears for $r_s = 6.02$. This value is in agreement with the Hartree–Fock results ($r_s \approx 5.4$) that were obtained before [4]. But comparing the RPA method ($r_s \approx 18.7$), it is smaller [4]. This difference is due to the correlation energy in RPA approximation [4].

5 Summary

The possibility of existence of the ferromagnetism phase inside the electron gas was investigated. The equations have been written at the low temperature limit. However, we can write the equations in general form. We used the relativistic one photon exchange interaction. Then we calculated the non-relativistic limit of interaction, because in the ultra-relativistic region the unpolarized states are energetically favorable. In this manner, the spin dependence of the Hamiltonian appears automatically. To obtain the thermodynamic quantities, we used the variational approach and found the minima of the thermodynamics potentials at various densities and temperatures. According to these results, the ferromagnetism phase can appear at low temperatures and high densities. We must note that the equations are valid only for high densities with constraint $k_f \ll m$. If we want to investigate very high densities, we must use the relativistic formalism without approximation. The coupling constant has an important role in the critical density and temperature for phase transition. The magnetic susceptibility at zero temperature becomes infinite at $k_f \longrightarrow \frac{m\pi}{\alpha}$. The results are in agreement with previous works and at $r_s = 6.02$ the inverse magnetic susceptibility of system becomes zero. Comparing the RPA results, our results are smaller and this is due to the correlation energy that is accounted for in the RPA method [4]. We also can obtain the critical exponents of the system by expanding the magnetization and magnetic susceptibility of the system near the critical point.

Acknowledgements We would like to thank Prof. M.V. Zverev for useful and important guidance. We also would like to thank the Research Council of the University of Tehran and the Institute for Research and Planning in Higher Education for the financial support and grants provided for us.

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